Batch Dynamic Algorithm to Find $k$-Cores and Hierarchies

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ABSTRACT
Finding $k$-cores in graphs is a valuable and effective strategy for extracting dense regions of otherwise sparse graphs. We focus on the important problem of maintaining cores on rapidly changing dynamic graphs, where batches of edge changes need to be processed quickly. Prior batch core algorithms have only addressed half the problem of maintaining cores, the problem of maintaining a core decomposition. This finds vertices that are dense, but not regions; it misses connectivity. To address this, we bring an efficient index from community search into the core domain, the Shell Tree Index. We develop a novel dynamic batch algorithm to maintain it that improves efficiency over processing edge-by-edge. We implement our algorithm and experimentally show that with it while remaining under re-computing from scratch.

1 INTRODUCTION
An important problem in graph analysis is finding locally dense regions in globally sparse graphs. In this work we consider the problem of finding $k$-cores [37, 41], which are maximal connected subgraphs with minimum degree at least $k$. This problem has seen significant attention given its efficiency [37] and usefulness across many domains [2, 17, 21, 22, 26, 27, 45].

Many practically important graphs from web data, social networks, and related fields are both large and continuously changing. The problem of maintaining core decompositions on graphs has been well studied [31, 38, 48, 49]. Existing approaches run in linear time in the size of the graph, which is theoretically optimal [48], and on many real-world graphs they maintain decompositions within milliseconds after edge changes. So, is the problem solved?

Unfortunately, these approaches only address half of the problem of returning a $k$-core [39]. $k$-cores are originally defined as connected subgraphs [41]. All of the application examples referenced above rely on or use connectivity. A core decomposition, on the other hand, provides a coreness value for every vertex that is, the largest value such that a vertex is in a $k$-core, but not in a $(k+1)$-core. Prior approaches have either ignored connectivity (which provides limited, but some insight e.g., [25]) or left the final step of finding components as a separate process. The main tool to address computing connectivity on cores, or a core hierarchy, has been independently proposed several times [5, 14, 16, 39] in different contexts, and concurrently developed in [32]. We introduce this index in the most basic setting, designed for $k$-cores on simple undirected graphs, and we call it the Shell Tree Index (ST-Index). This index supports queries to extract the cores a vertex is in along with the full core hierarchy of a graph.

Example Problem. Consider the problem of managing a social network. First, given a user, we wish to recommend friends to them that are well connected in their part of the graph: this is a vertex and coreness query. Second, we want to detect structural changes, for example sybil attacks [12] from new fake accounts: this is a hierarchy query. Figure 1 shows the core hierarchy of the LiveJournal graph [46] and how far apart different dense regions are. For both query examples, we want results in tens of milliseconds to either prepare a webpage or mitigate an emerging attack.

In this example scenario, a state-of-the-art core decomposition system is put in place, which provides coreness updates quickly after graph changes. The two goals above require information about specific cores. If certain vertices achieve higher coreness values, this does not inform whether a new region is created. Furthermore, unless there is only one dense region, it will not enable useful recommendations. Instead, we need systems and algorithms that can quickly and effectively return cores themselves along with their full hierarchies.

Approach. The ST-Index builds on the laminar nature of cores. For $k' < k$, every $k$-core is contained within some $k'$-core, naturally forming a tree. Each node in this tree corresponds to the shell of the core, that is vertices which are not in any higher core. Coupled with a reverse map, a core can be efficiently returned by traversing

Figure 1: The core hierarchy for the LiveJournal social network graph. Tracking dense regions behavior over time is important for understanding structural changes, and extracting the vertices within a dense region is important for almost all known $k$-core applications.

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the subtree staying below the desired \( k \) value. The core hierarchy is the tree.

We build the tree by first identifying regions of the graph where the cores are the same, known as subcores, and then forming a directed acyclic graph (DAG) with each subcore as a node. Starting from the highest \( k \) values, we process nodes in the DAG upwards, merging and moving them to form a tree.

The only known prior maintenance approach, operating for attributed graphs and used as part of solving community search, is given in [14]. We first port this maintenance approach to the case of \( k \)-cores on standard graphs and use that as our edge-by-edge baseline. Given an edge change, it maintains the ST-Index by either merging or splitting nodes on paths to the root. Concurrent with this work, [32] builds on [14]’s approach by batching operations on the tree.

In real-world graphs there is significant variance in the rate of change. As such, batch dynamic algorithms that can reduce the total work when operating on batches are desired [11, 35]. We provide a batch dynamic algorithm to maintain cores themselves, starting from core decompositions. We do this by maintaining the subcore DAG used during construction. After a batch of changes, we revisit each node in the DAG that was modified and re-compute any subcore changes. Any DAG changes are then pushed into the tree, temporarily turning the tree back into a DAG. We then traverse from the sink upwards, correcting the tree.

**Contributions.** In addition to bringing the ST-Index from the community search domain into the direct, \( k \)-core domain, we prove efficiency properties on the ST-Index. Our main contributions are:

1. A subcore DAG based ST-Index construction algorithm
2. A batch dynamic algorithm to maintain ST-Index that reduces the work of edge-by-edge updates
3. An experimental evaluation on real-world graphs that show with both our edge-by-edge and batch algorithms, ST-Index is suitable for interactive use

The remainder of this paper is structured as follows. In § 2 we describe the related work. In § 3 we formally describe our model and problem. In § 4 we present ST-Index. In § 5 we provide our algorithm to compute ST-Index from scratch. In § 6 we explain how to maintain ST-Index for dynamic graphs and introduce our batch algorithm. In § 7 we experimentally evaluate our implementations, and in § 8 we conclude.

### 2 RELATED WORK

\( k \)-cores were introduced independently in [37, 41], [37] additionally provided a peeling algorithm that uses bucketing to run in \( O(n+m) \). The main strategy for computing \( k \)-cores has remained roughly the same since then: iteratively peeling the graph, or excluding vertices with too low of degrees, until all degrees are \( k \).

For maintenance, [31] and [38] independently proposed Traversal, which limits consideration of vertices around an edge change if they provably cannot update values. [38] defines the notion of subcores and purecores, variants of which are used in all known maintenance algorithms to limit considered subgraphs. [49] proposed Order, which is the current state-of-the-art and maintains a peeling order, instead of coreness values directly, using an order-statistic treap and a heap. Parallel approaches have relied on identifying a set of vertices that can be independently peeled [1, 3, 23, 24], [4, 48] provide batch algorithms that reduce work as multiple edges are processed simultaneously.

All of the above focus on computing the coreness values for vertices. In fact, the lack of focus on connectivity has, in some cases, resulted in later work redefining cores to not include connectivity (e.g., [36]) which limits their usefulness.

Numerous other targets, similar to cores, have been proposed [36], [19, 50] develop weighted extensions to cores, [33] uses core concepts to reinforce connections within networks, [20] proposes notions of cores for multilayer networks, and [47] ensures vertices in core-like regions are also relatively cohesive given their neighbors. In cases where the cores are used for downstream algorithms, returning the actual (connected) vertices is identified as crucial and algorithms are built to support such queries [34].

Community search [9, 43] is a more general problem for returning a connected set of vertices in a community based on a seed set. The community is commonly defined with a minimum degree measure[15]. In this case, if the query consists of a single vertex, community search can return exactly a core. For this reason, we pull from the field of community search to develop ST-Index. [5] proposed the first known shell tree index. It does not support efficient queries, as it creates additional vertices for each coreness level that must be addressed. [39] identifies the same problem that we address—cores require connectivity—and proposes a shell tree-like index with a static construction in the more general nuclei framework, but leaves out maintenance. [14] operates on attributed graphs and extends [39]’s approach and [5]’s index with incremental and decremental algorithms, but without batch algorithms. We port this approach to the problem of cores and use this as our baseline. Concurrent with this work, [32] provides a batch algorithm that is based on [14] and batches changes to the tree directly, without the use of a DAG.

### 3 PRELIMINARIES

A graph \( G = (V, E) \) is a set of vertices \( V \) and set of edges \( E \). An edge \( e ∈ E \) represents the connection between two distinct vertices \( u, v ∈ V \), \( e = \{u, v\} \). We denote \(|V|\) by \( n \) and \(|E|\) by \( m \).

We use \( Γ(v) \) to represent the neighboring edges of \( v ∈ V \). The degree of \( v \in V \) is \( |Γ(v)| \). For directed graphs, \( Γ^\text{in} \) represents edges ending at the given vertex and \( Γ^\text{out} \) represents edges leaving a vertex. If the graph is ambiguous, we use \( Γ_G \) for graph \( G \). The neighborhood of a vertex set \( S \subseteq V \), \( Γ(S) \), represents the vertices and edges connected to \( S \), that is it is the subgraph induced by \( S \) and all neighbors of vertices in \( S \).

**Dynamic Graph Model.** We consider graphs that are changing over time, known as dynamic graphs. An edge change is a tuple \((c, v, e)\) consisting of a direction \( c \), a vertex \( v \in V \), and an edge \( e ∈ E \). A dynamic graph is then an infinite turnstile stream of edge changes \( S \), where time is the position in the stream. At any point in time an undirected graph \( G^t \) can be formed by applying all edge changes until \( t \), starting from an empty graph.

In this model, the timestamp of edges received is not preserved and not used by the algorithm. An algorithm that does take into consideration timestamps is called a temporal algorithm, and can either be dynamic or static.
Definition 3.1. Let $\mathcal{A}$ be a graph algorithm with output $\mathcal{A}(G)$. Then $\mathcal{A}_t$ is a dynamic graph algorithm if, for some times $t$ and $t'$, with $t < t'$,

$$\mathcal{A}_t(G', \mathcal{A}(G'), \mathcal{A}_{t+1}' \mathcal{A}(G'), \mathcal{A}_{t+1}', S[t, t']) \equiv (\mathcal{A}(G'), \mathcal{A}_{t+1}' \mathcal{A}(G'), \mathcal{A}_{t+1}', S[t, t']),$$

where $\mathcal{A}_t'$ contains algorithm state at $t$ and $S[t, t']$ are the edge changes in $S$ from $t + 1$ to $t'$.

We call an incremental algorithm a dynamic graph algorithm which can only handle edge insertions and a decremental algorithm one which can only handle edge deletions. A batch dynamic algorithm can handle $\Delta > t + 1$. Our batch algorithm, described in Section 6, has an additional state bound by the size of the graph.

Cores. We provide a brief background on cores.

Definition 3.2. Let $G$ be a graph and $k \in \mathbb{N}$. A $k$-core in $G$ is a set of vertices $V'$ which induce a subgraph $K = (V', E')$ such that: (1) $V'$ is maximal in $G$; (2) $K$ is connected; and (3) the minimum degree is at least $k$, $\min_{v \in V'} \deg v \geq k$.

Figure 2 shows an example graph and its cores. There are two separate $k$-cores, one with vertices 1 through 4 and the other with vertices 7 through 10. If all vertices with less than a degree 3 are iteratively removed, the remaining graph consists of those two separate connected components.

Definition 3.3. Let $G = (V, E)$ be a graph and $v \in V$. The coreness of $v$, denoted $\kappa_v$, is the value $k$ such that $v$ is in a $k$-core but not in a $(k + 1)$-core.

Definition 3.4. Let $G = (V, E)$ be a graph. The $k$-core number of $G$, denoted $\rho_G$ and shortened to $\rho$, is given by $\rho = \max_{v \in V} \kappa_v$.

Problem Statement. We consider the problem of efficiently supporting core and coreness queries on a dynamic graph stream. Let $k \in \mathbb{N}$ and $u \in V$.

- The coreness query $\mathcal{K}(u)$ returns $\kappa[u]$.
- The core query $C(u, k)$ returns the vertices of the $k$-core subgraph that contains $u$.
- The hierarchy query $\mathcal{H}$ returns the hierarchical structure of the cores as a tree, with the root as the 0-core.

Prior work in the context of cores has focused only on supporting $\mathcal{K}$ queries on dynamic graphs. Unfortunately, this prevents many of the applications of $k$-cores which rely on extracting dense regions of a graph.

4 SHELL TREE INDEX

In this section we present the Shell Tree Index, ST-Index, which is able to efficiently return cores for different vertices: its runtime is asymptotically the size of the result and its space is linear in the number of vertices. This index has been independently developed several times [5, 14, 16, 32, 39] in different contexts. We present the index here for completeness. We will address how to construct the index in Section 5 and how to maintain it in Section 6.

$\mathcal{K}(u)$ queries, or coreness queries, can be efficiently returned using an array of size $n$. We therefore focus on $C$ and $\mathcal{H}$ queries.

Lemma 4.1 ([38]). Cores form a laminar family, that is every pair of cores are either disjoint or one is contained in the other.

Proof. We want to show that for every two cores $K_1$ and $K_2$, $K_1 \cap K_2$ is exactly one of $\emptyset$, $K_1$, or $K_2$.

Let $K_1$ and $K_2$ be two cores of $G$, with corresponding $k$ values $k_1$ and $k_2$. Suppose $K_1 \cap K_2 \neq \emptyset$, implying $K_1$ is connected to $K_2$. Note that $k_1 \neq k_2$, otherwise $K_1 \cup K_2$ is a $k_1$-core, invalidating maximality. Let $k_1 < k_2$. Suppose $\exists o \in K_2$ such that $o \notin K_1$. $o$ must be connected in $K_2$, and so there exists a path from $K_1$ to $o$ with minimum degree at least $k_2$. Let $K'_o$ be a subgraph that includes $K_1$ and the path to $o$. Then, $K'_o$ is a $k_1$-core and larger than $K_1$, invalidating maximality.

Definition 4.2. Let $G = (V, E)$ be a graph and $K \subseteq V$ a $k$-core in $G$ for some $k \in \mathbb{N}$. Then $S$ is a $k$-shell if $S = \{v \in K : \kappa_v = k \}$.

Note that the shell is disconnected, however it is a subset of a connected core. This means that the traditional approach of using coreness values to compute the shell does not work. We address shell computation later in Section 5, using subcores.

A shell tree $T$ is at the heart of the ST-Index. We call the vertices of $T$ tree nodes, to distinguish from the vertices in $G$. Each node has two additional pieces of data associated with it: a $k$ value and a set of vertices (in $G$). $T$ is built as follows. A root node is made with $k = 0$ and a vertex set of isolated vertices (those with $\deg v = 0$). Next, nodes are made in $T$ for every $k$-shell. Its $k$ attribute is set to $k$ corresponding to the shell and its vertex list is set to the vertices in the $k$-shell. An edge is created in $T$ by linking $k$-shells, following Lemma 4.1. An example shell tree is shown in Figure 3. The ST-Index consists of $T$ and a map $M$, mapping $v \in V$ to the appropriate node in $T$.  

Figure 2: An example graph and its cores. Note that there are two separate 3-cores.

Figure 3: The shell tree for the graph shown in Figure 2. On the left side are the $k$-shell values, and on the right side are the contained vertices. Each directed edge indicates inclusion of the deeper cores.
Lemma 4.3. The shell tree is a directed, rooted tree.

Proof. Suppose a tree node \( u \), corresponding to core \( K_u \), has two in-edges. By definition 4.2, each parent corresponds to a unique \( k \)-shell. Consider the two corresponding cores, \( K_1 \) and \( K_2 \). They both include \( K_u \), yet are distinct, and so they have non-trivial overlap contradicting Lemma 4.1. The root is defined with \( k = 0 \).

Lemma 4.4. The out-degree of a non-root tree node with no corresponding vertices in the shell tree can be at most 1.

Proof. Let the tree node with no corresponding vertices be at level \( k > 0 \) with out-degree at least 2. Then, there are two distinct cores at \( k + 1 \) (not necessarily shells), and one core at \( k \). The two cores at \( k + 1 \) must be disconnected by construction.

However, because the tree node has no corresponding vertices, we know that every vertex in the \( k \)-core is also in a \((k + 1)\)-core. Furthermore, the \( k \)-core is connected. Hence, it is not possible for the two cores at \( k + 1 \) to be disconnected.

Lemma 4.5. Let \( G = (V, E) \) be a graph with \( n = |V| \). The number of nodes in the shell tree is at most \( n + 1 \) and edges is at most \( n \).

Proof. By Lemma 4.4, each node in the tree (besides the root) must have at least one vertex. As there are at most \( n \) vertices, the size of the tree is at most \( n + 1 \). By Lemma 4.3, we know it is a tree, and so with at most \( n + 1 \) nodes it has at most \( n \) edges.

Queries on ST-Index. The three queries, \( \mathcal{K}(u) \), \( C(u, k) \), and \( \mathcal{H} \) are returned as follows.

- \( \mathcal{K}(u) \) follows the map \( M[u] \) to the shell tree node \( n \), and then returns the \( k \)-core for \( n \).
- \( C(u, k) \) runs a tree traversal that stays above the level \( k \).
- \( \mathcal{H} \) returns the tree nodes and attributes directly.

Efficiency of ST-Index. We next address the efficiency of queries on ST-Index.

Theorem 4.6. \( C(u, k) \) queries on ST-Index run in \( O(|C(u, k)|) \) and correctly return the \( k \)-core.

Proof. First, we show correctness. Let \( C^* \) be the core for \( C(u, k) \), that is \( C^* \) is a \( k \)-core and \( u \in C^* \). The traversal will cover all vertices in the subtree containing \( u \) at level \( k \) and higher. By Lemma 4.1, we know all denser cores are fully contained in the desired \( k \)-core. By Lemma 4.4, we know that any split will occur in an explicit tree node with vertices in the resulting shell. So, this split will be captured by the tree traversal. As such, all vertices in the tree nodes traversed with values \( k \) or more exactly form the \( k \)-core.

Let down represent higher \( k \) values in the tree. Next, we show efficiency. Every downward link in the subtree needs to be fully explored, and there are no nodes with overlapping vertices in the tree. Once a downward traversal occurs, there is no need to check parents. When traversing upwards, all children except the previous one will be explored downwards. In each case every node is visited exactly once and all of its associated vertices are enumerated once and are part of the returned core.

As ST-Index is a tree, whether to traverse to the parent can be decided based on whether the parents’ value is lower than \( k \). This will result in one additional operation. As such, the runtime is \( O(|C(u, k)|) \) and efficient.

Theorem 4.7. The ST-Index takes \( O(n) \) space.

Proof. The ST-Index consists of a map of size \( n \) between vertices and tree nodes, along with the shell tree itself. By Lemma 4.5, the tree has at most \( n + 1 \) nodes and \( n \) tree edges. Each tree node may have vertices, but there are no redundant vertices. So, the size is \( O(n + n + n + n) = O(n) \).

The shell tree itself contains the hierarchy of cores and shells, and so returning ST-Index efficiently resolves \( \mathcal{H} \) queries.

5 COMPUTING THE ST-INDEX

Computing (and maintaining) the ST-Index hinges on building (and maintaining) the shell tree. We propose a subcore directed acyclic graph, that provides the link between core decompositions and the shell tree. In this section we describe how to compute the ST-Index from scratch using the subcore DAG.

This problem is broken into three parts: computing coreness values, subcore DAG, and the shell tree.

Computing Coreness Values. Computing coreness values has been well studied on graphs [10, 37]. The most direct approach, known as peeling, starts by keeping an array of vertex degrees. It then moves up through coreness values, removing vertices with insufficient degree and recording when they are removed. This is efficient, running in \( O(n + m) \), when using buckets [37]. We refer the reader to [36] for a survey.

Computing the Subcore DAG. Next, we introduce the subcore directed acyclic graph (DAG), which is used to bridge between coreness values and cores.

Definition 5.1. Let \( G \) be a graph. A subcore is a subgraph \( C \) such that \( (1) C \) is maximal \((2) \forall v \in C, \kappa[v] = k \) for some \( k \in \mathbb{N} \) and \( (3) C \) is connected.

Subcores were introduced in [38] to limit the region that may have coreness values change on graph changes. Figure 4 shows an example graph with cores and subcores.

Observation 5.1. Subcores are disjoint, by maximality of cores and property (2), and so the number of subcores is bound by \( n \).
The subcore DAG size is bound by $G$.

Proof. Each vertex in the subcore DAG corresponds to a connected subgraph in the graph, and every edge in the DAG is a directed edge that results from contracting all vertices in each subcore. Contraction only removes edges and vertices, and no new edges or vertices are added.

Observation 5.2. The subcore DAG is not a tree. Consider a 3-clique and a 4-clique, connected via an edge, and both connected to another vertex. This forms a directed triangle in the DAG.

The process of building the subcore DAG is shown in Algorithm 1. This algorithm performs a breadth-first search (BFS) for each vertex. The search is constrained to stay within a $\kappa$ level, and DAG edges are emitted on graph edges that leave $\kappa$ levels. Efficient connected components algorithms, e.g., [42], could be used instead.

**Lemma 5.3.** Algorithm 1 runs in $O(n + m)$.

Proof. From lines 6–11, inside the internal BFS, each vertex will be visited once. Inside, each edge will be visited once. Finally, the entire BFS will only start from unvisited vertices. For lines 12–14, each vertex and edge will again be visited, resulting in $O(n + m)$ work.

### 5.1 Building the Shell Tree

Given a subcore DAG and $\kappa$ values, we can compute the shell tree. Our algorithm starts with the DAG and modifies it as it moves from the sinks upwards (towards lower $\kappa$ values), using a max-heap. Each processed vertex: 1) identifies neighbors that are at its $\kappa$ level, and merges itself with them; 2) sets a single node that is an in-neighbor with the closest $\kappa$ value as the tree parent; and 3) moves all other in-edges to the identified parent, ensure it becomes a tree. The details are presented in Algorithm 2.

**Lemma 5.4.** Algorithm 2 correctly builds the shell tree.

Proof. We argue that after running Algorithm 2, each node will exactly contain the shell. First, a node needs to contain all connected subcore DAG nodes at the given $\kappa$ value. Second, it cannot have additional nodes merged with it. We argue correctness via induction on $\kappa$. At the highest $\kappa$ level, by the DAG properties, we know the tree nodes connected to the sink are shells and valid. Now, consider a tree node with $\kappa$ and assume nodes at $\kappa' > \kappa$ are valid. The node is formed by merging DAG nodes at the same level, which are all connected. Any connectivity that is not at level $\kappa$ will be preserved by moving edges to the node’s parent. By Lemma 4.1, we know that any DAG neighbors that it is connected to will also be connected to the parent, and so the new tree node is valid.
6 MAINTAINING THE ST-INDEX

In this section, we show how to maintain the ST-Index on a graph stream. The objective is to develop a batch dynamic algorithm $\mathcal{A}_3$ that will output the shell tree ST-Index, while having a small internal state $\mathcal{A}_3^\ast$ and a quick runtime with low variability.

6.1 Maintaining Coreness

We refer the reader to [18, 31, 38, 49] for algorithms to maintain $k$. These approaches (and similarly ST-Index) extend to trusses [8] and other nuclei [40] by use of a hypergraph [19]. For our experiments we implemented and use Order [49], the state-of-the-art decomposition maintenance algorithm.

For notational convenience, consider a time $t$. Let $G^\ast$ denote $G_t^\ast$ and $G^+$ denote $G_t^+$. Let $k^-$ denote the $k$ values in $G^-$ and $k^+$ denote the $k$ values in $G^+$.

We take advantage of the following crucial property of coreness values on graphs: the subcore theorem.

THEOREM 6.1 ([38]). Let $\{u, v\}$ be an edge change. Suppose $\kappa_{G^-}\{u\} \leq \kappa_{G^-}\{v\}$. Then, only vertices in the subcore containing $u$ may have $k$ values change in $G^+$, and they may only change by 1 (increase by 1 for insertion, decrease by 1 for deletion.)

6.2 Single Edge Maintenance Algorithm

The main idea for maintaining the ST-Index edge-by-edge is to first break apart any core or shell that was increased and then repair the tree by merging together the paths from the endpoints. For deletions, a map is made that determines where, after a core is split, it could return to in the tree. Then, the path from the core to the root is traversed and any potential split is determined. Our algorithm shares many similarities to the community search algorithm of [14]. Our algorithm addresses cores instead of the more general community search problem on attributed graphs. Specifically, it does not need to support queries involving subsets of vertices. We refer to this approach as SingleEdge. We describe insertions in detail—deletions are similar but split nodes [14].

Let $K$ be the tree node that has a lower $k$ value given an edge insertion. We first check if all of $K$’s vertices leave. If so, we move $K$ down and merge its children with connected subcores. Next, we iterate through the moved vertices and identify if they are connected to a shell tree node at level $k + 1$. If so, we merge those shell tree nodes together. If not, we create a new tree node for the moved vertices. Then, we walk up the tree from both endpoints and, starting at level $k + 1$, begin merging all visited vertices. The algorithm is presented in Algorithm 4, with merge paths presented in Algorithm 4. A visual depiction is given in Figure 6.
We now present our batch maintenance algorithm. First, we maintain the subcore DAG by iterating every node in the tree and every node in the subcore DAG. There are two main parts to maintaining the subcore tree in the subcore batch algorithm. First, we maintain the subcore DAG by iterating over changed vertices and recomputing any subcores, creating and merging subcores (locally) as appropriate. Second, we need to maintain the ST-Index given the DAG changes. To do this we begin by making all of the changes propagate forward to the tree. Any deleted DAG node results in deleting the reference from the subcore tree, any newly empty tree nodes are deleted, and any new DAG nodes and their connections are added to the tree. The tree is now no longer a DAG. We then run the heap-based Algorithm 2 to finish turning the modified structure back into a tree. The tree is now no longer a DAG. We then run the heap-based Algorithm 2 to finish turning the modified structure back into a tree. In the worst case this can be the runtime of Algorithm 2. However, note that the BFS on subcores is limited to modified subcores. As such, empirically we run faster than re-computing from scratch, as shown in the following Section 7.

**6.3 Batch Maintenance**

We now present our batch maintenance algorithm. First, we present the opportunity for reducing work by providing an example. In Figure 7, we show the graph before and after the batch.

The idea is to keep the subcore DAG in memory and use it to update the subcore tree. This can naturally be combined with SingleEdge to provide a hybrid approach, moving between the two based on a batch size. We maintain an additional pointer between every vertex in the tree and every vertex in the subcore DAG. There are two main parts to maintaining the subcore tree in the subcore batch algorithm. First, we maintain the subcore DAG by iterating over changed vertices and recomputing any subcores, creating and merging subcores (locally) as appropriate. Second, we need to maintain the ST-Index given the DAG changes. To do this we begin by making all of the changes propagate forward to the tree. Any deleted DAG node results in deleting the reference from the subcore tree, any newly empty tree nodes are deleted, and any new DAG nodes and their connections are added to the tree. The tree is now no longer a DAG. We then run the heap-based Algorithm 2 to finish turning the modified structure back into a tree. During this process we maintain the reverse vertex maps. Unlike SingleEdge, our batch approach naturally covers deletions identically to insertions and both insertions and deletions can be mixed inside of batches. This is due to handling both endpoints of an edge change, instead of only the endpoint with a lower \( \kappa \) value at some point in time. The approach is shown in Algorithm 5. Following the example in Figure 7, we show the saved work between SingleEdge and Batch in Figure 8 (next page).

Our runtime is the cost of Algorithm 2 plus the cost of a BFS over each modified subcore. Correctness follows from Algorithm 2 as we maintain the built data structures and operations. In the worst case this can be the runtime of Algorithm 2. However, note that the BFS on subcores is limited to modified subcores. As such, empirically we run faster than re-computing from scratch, as shown in the following Section 7.

**Algorithm 5:** The Batch algorithm.

**7 EMPIRICAL ANALYSIS**

In this section we perform an experimental evaluation of our approach to demonstrate that it is able to provide core queries on rapidly changing real-world graphs.

**Environment.** We implemented our algorithm in C++ and compiled with GCC 10.2.0 at 03. We ran on Intel Xeon E5-2683 v4 CPUs at 2.1 GHz with 256 GB of RAM and CentOS 7. To perform coreness maintenance, we implemented Order [49]. Any coreness maintenance approach can be used in its place. We include all memory allocation costs in our runtimes. We use a hash map of vectors to store the graph, and store both in- and out-edges. We ran five trials for each experiment and show the results from all trials.

**Baseline.** As our baseline, we implemented the non-batch maintenance approach from [14], which we ported to the case of computing cores on graphs (see Section 6.2). We refer to this as SingleEdge. When operating on a batch, SingleEdge runs independently for each edge change. Insertions and deletions can therefore easily be mixed. We only show results with insertions as they are the harder case [14] and there are few known benchmark datasets with frequent deletions.
Figure 8: Following the example in Figure 7, we show the tree changes processing with SingleEdge compared with our batch approach. The cost is an increase in memory to store the subcore DAG and unnecessary work if a modified subcore does not significantly change.

Table 1: Graphs used with $n, m$ in millions.

| Name     | $n$, $m$ | DAG $n$, $m$ | $|T|$ |
|----------|----------|---------------|------|
| Ar-2005  | 22, 640  | 12, 47       | 28 K |
| Orkut    | 3, 117   | 1, 22        | 254  |
| LiveJ    | 4, 35    | 2, 12        | 2 K   |
| Pokec    | 2, 22    | 1, 5         | 54    |
| Patents  | 4, 17    | 2, 4         | 4 K   |
| BerkStan | 0.7, 7   | 0.2, 0.8     | 2 K   |
| Google   | 1, 4     | 0.4, 1.2     | 5 K   |
| YouTube  | 1, 3     | 1, 2.5       | 140   |

Figure 9: The ST-Index construction time, broken down into DAG construction and Tree construction.

Datasets. The graphs that we evaluate with are benchmark graphs that are representative of real-world graphs from a variety of domains and with different properties. We downloaded them from SNAP [29] (excluding Ar-2005, downloaded from [7]). The graphs we use are given in Table 1. We cleaned the data by removing self loops and duplicates edges and treated graphs as undirected. We randomized the edge order, simulating a graph stream, and performed our experiments by first removing random edges and next inserting them.

Experiments. Our main experimental goal is to evaluate the real-world feasibility of our approach on modern graphs and systems with highly variable and large batch sizes.

First, we show the index construction time for Batch. The results are shown in Figure 9. In all cases building the tree is more expensive than building the DAG. The overall runtime reinforces the need for dynamic algorithms as for large graphs, such as Orkut, the DAG construction takes around 90 seconds and the tree construction takes around 330 seconds.

Next, we want to show that ST-Index is a useful index for cores. We report the query times for $C$ in Figure 10 and $H$ in Figure 11 on ST-Index. For $C$, we performed queries from 1000 randomly sampled vertices with uniformly random $k$-values such that the vertex is in a $k$-core. For all graphs, all cores are returned in under one second with many in the tens of milliseconds. Given that our query is efficient the runtime largely consists of copying memory. The denser the core the faster the return tends to be, as there are fewer vertices to copy out. In many cases, the runtimes are fast enough to be used for interactive applications, e.g., in web page content. For $H$, we report the time to build and return the full hierarchy, including each node at each level. This is under 10 seconds for all graphs, showing that full hierarchies can be used for interactive time applications.
Finally, we maintained cores for 100 batches of different batch sizes for each graph. The results are shown in Figure 12. In all cases, when batch sizes are large Batch remains below both FromScratch and SingleEdge. For a batch dynamic algorithm, we are looking for the region below re-computing from scratch and below single-edge algorithm. In some graphs, such as Pokec and Patents, it is not a large region, however in all graphs it exists and provides significant improvements. Future work involves combining the DAG construction and maintenance with the direct tree maintenance to achieve an effective hybrid approach, achieving the lower of the all of the curves. Note that these are log-log plots, and so even for Patents our batch approach is 2x faster than re-computing from scratch at batch sizes of one million.

8 CONCLUSION

We focus on the important but overlooked problem of returning cores, as opposed to coreness values. We consider both core queries, which return a $k$-core, and hierarchy queries, which return the full core hierarchy. Our approach applies beyond $k$-cores to other arbitrary nuclei, such as trusses.

We develop algorithms around a tree-based index, the ST-Index, that is efficient and takes linear space in the number of graph vertices. We provide an algorithm to construct the ST-Index using a new approach based on a subcore DAG. We design and implement a batch maintenance algorithm for ST-Index that uses the same subcore DAG and can handle variable and high batch sizes. We show that our approach is able to run faster than edge-by-edge approaches on rapidly changing graphs and can return cores and hierarchies fast enough for interactive use.

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